Analytical Approach to Vibration Analysis Of the Wheel-rail contact

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Abstract

Wheel/rail contact simulation is one of the most complicated problems in the modeling of railway vehicles. The wheel/rail interaction plays a unique role in rail vehicle dynamics. In this paper, the dynamic response of the wheel on irregular rail track is analyzed with analytical approach using the method of Multiple Scales (MMS). The Hertzian contact theory is used to obtain the relationship between normal contact force and the displacement of the mass center of the wheel. Analytical approach is expanded for performance of train's wheel travelling on the rail. To validate the method presented in this paper, responses of the model using MMS method are compared with the results obtained from the Runge–Kutta numerical solution. Finally effects of the wheelset preload on response frequency have been studied.

Keywords: *Wheel-rail contact, Hertz elastic contact, Method of Multiple Scales, Runge–Kutta method*

1. Introduction

Surfaces of engineering components are routinely subjected to contact loading, where large stresses are applied over highly localized area. Many researchers have studied different parameters and their influence on the dynamic behavior of rotary systems. Nielsen presented a nonlinear wear model for the contact of a cylinder rolling over a periodically varying surface [1]. His model was a pure contact one and the dynamics of the cylinder and surface were neglected. El-Sayed obtained the stiffness of bearings using the Hertzian contact model [2]. He also calculated the total deflection of inner and outer races due to an applied force. Pandiyarajan determined the contact stress of large diameter ball bearings using analytical and numerical methods [3]. In analytical method the contact stress was found out using the Hertzian elliptical contact theory. Also, finite element analysis performed to predict contact pressure in ball and raceway.

Vibration analysis is highly important in mechanical systems, especially in rotary machines. The effect of contact forces on the vibration behavior of rotary components has been known for many years but a fully satisfactory explanation of the phenomenon has not yet been found. Aram et al. used analytical approach to obtain vibration characteristics of spherical ball on the plate [4]. Xiaoa Used three methods to determine the natural frequency of undamped free vibration of a mass interacting with a Hertzian contact stiffness [5]. The exact value was determined using the first integral of motion. The harmonic balance method was used on a transformed equation for an approximate solution, and the multiple scales method was used on an approximate equation.

The Hertz theory has been the only analytical solution for the 3D-smooth normal contact problem. Its simplicity and reliability entails its widespread utilization in modeling of the wheel/rail contact [6]. Transfer of load from the train to the track is essentially through the rail-wheel contact. Basically the train is moving on elastic support. While the train moves in the longitudinal direction, the train and track vibrate in all possible directions. However, vertical vibrations are the most predominant amongst all, and therefore this mode of vibration is the most studied [7]. Because dimensions of the contact area for wheel and rail is much smaller than the dimensions and the radii of curvature of the bodies in contact, Hertzian contact theory (HCT) can be applied for modeling of contact between wheel and rail. The HCT leads to an elliptical contact area and a semi-ellipsoid contact pressure distribution in the contact region. The

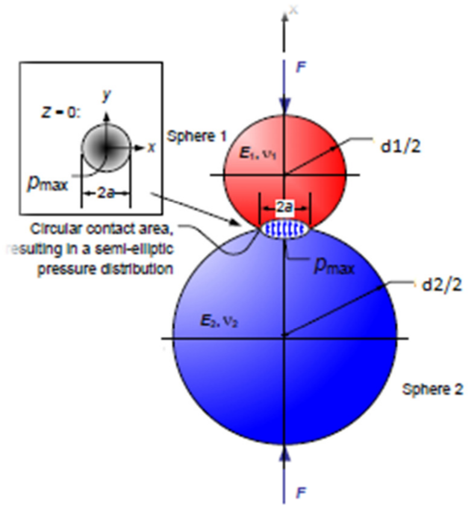

Fig1. Two spheres held in contact by means of the compressive force

application of the Hertz contact theory in rail-wheel contact problems can also be found in the recent literature. Yan and Fischer analyzed the Hertzian contact problem in wheel rail applications [6]. Comparing the Hertzian results with those obtained from three-dimensional finite element models they verified the Hertzian contact theory to wheel/rail contact problems. Baeza et al. studied the wheel/rail dynamic response caused by a geometric irregularity, by means of Hertzian and non-Hertzian contact models [8].

The objective of this study is to develop an analytical model for the vibration analysis of a wheel on an irregular rail track under a compressive force. The Hertzian contact theory is used for obtaining the force/ displacement relation. Also, the Method of Multiple Scale is implemented to evaluate analytically the time response of the system.

2. Theory and formulation

2-1. Hertzian contact

The Hertzian contact refers to two nonconforming bodies in contact, where the area of contact is significantly smaller than the dimensions of the bodies and the radii of surface curvatures [9].
Figure 1 shows two spheres held in contact by means of the compressive force F.

In order to model the contact between balls, the Hertzian contact theory is used. In this theory, principal stresses in x, y and z directions are introduced as [10]:

$$\sigma_z = \frac{-p_{max}}{1+{\tau_a}^2} \tag{1}$$

$$\sigma_x = -p_{max}\left[\left[1-|\tau_a|tan^{-1}\times\frac{1}{|\tau_a|}\right](1+\nu)-\frac{1}{2(1+{\tau_a}^2)}\right] \tag{2}$$

$$\sigma_y = \sigma_x \tag{3}$$

where

$$p_{max} = \frac{3F}{2\pi a^2} \tag{4}$$

$$\tau_a = \frac{z}{a} \tag{5}$$

$$a = K_a\sqrt[3]{F} \tag{6}$$

$$K_a = \left[\frac{3}{8}\frac{\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}}{\frac{1}{d_1}+\frac{1}{d_2}}\right]^{1/3} \tag{7}$$

$$\begin{aligned}\epsilon_x &= \frac{1}{E}\left(\sigma_x-\nu\left(\sigma_y+\sigma_z\right)\right)\\ \epsilon_y &= \frac{1}{E}\left(\sigma_y-\nu\left(\sigma_x+\sigma_z\right)\right)\\ \epsilon_z &= \frac{1}{E}\left(\sigma_z-\nu\left(\sigma_x+\sigma_y\right)\right)\end{aligned} \tag{8}$$

Integrating the z component of the strain tensor through the radius of the sphere, displacement of the center of the sphere, due to the compressive force, can be calculated as:

$$\delta = \left|\int_0^R \varepsilon_z dz\right| = \frac{p_{max}a}{E}\int_0^{d_1/2a}\left(\frac{1}{1+\tau_a{}^2}\right)d\tau_a + \frac{2\nu p_{max}}{E}\int_0^{d_1/2a}\left(\left[-1 + |\tau_a|tan^{-1}\left(\frac{1}{|\tau_a|}\right)\right] \times (1+\nu) + \frac{1}{2(1+\tau_a{}^2)}\right)d\tau_a \quad (9)$$

Substituting Eq. (5) into Eq. (9), we have:

$$\delta = \frac{p_{max}}{E}a.C = \frac{3F}{2\pi aE}.C \quad (10)$$

in which C is a function of the geometric and mechanical properties of the system, and can be evaluated from Eq. (9). It is noted that the numeric value of C is obtained after determining the sample's dimensions. From Eqs. (6) and (10), the relation between force and displacement is obtained as:

$$F = K'\delta^{3/2} \quad (11)$$

In which K' is introduced as follows:

$$K' = \left(\frac{2\pi EK_a}{3C}\right)^{3/2} \quad (12)$$

2-2. Vibration modeling of the system

It can be realized from Eq. (11) that the relationship between force and deflection is nonlinear. Modeling the system with a mass and a nonlinear spring, the vibration equation is obtained as:

$$m\ddot{x} + K'x^{3/2} = F \quad (13)$$

But when the wheel rotates on the sinusoidal rail surface like figure 2, the vibration equation can be rewrite as follow:

$$m\ddot{x} + K^{'}(x - \varepsilon^2 a^2 sinVt)^{1.5} = F \quad (14)$$

In which m is the equivalent mass and x is the vertical displacement of the center of the wheel and $\varepsilon^2 a^2$ is amplitude of rail irregularities. We assume that the amplitude of rail irregularities is small. Equation (14) can be rewrite in this form:

$$\ddot{x} + \lambda^2(x - \varepsilon^2 a^2 sinVt)^{1.5} = {}^F/_m \quad (15)$$

Where

$$\lambda^2 = \frac{1}{m}\left(\frac{2\pi EK_a}{3C}\right)^{3/2} \quad (16)$$

If we define parameter $x = z + \varepsilon^2 a^2 sinVt$ therefore equation (15) change like this

$$(\ddot{z} - \varepsilon^2 a^2 V^2 sinVt) + \lambda^2(z)^{1.5} = {}^F/_m \quad (17)$$

$$\ddot{z} + \lambda^2(z)^{1.5} = \varepsilon^2 a^2 V^2 sinVt + {}^F/_m \quad (18)$$

It is obvious that by applying the compressive force, the static equilibrium position is displaced from z = 0 to z = z*. In order to solve Eq. (18), we first obtain Taylor expansion of z1.5 about z*:

$$z^{1.5} = (z^*)^{1.5} + \frac{3}{2\times 1!}(z^*)^{0.5}y + \frac{3}{2^2\times 2!}(z^*)^{-0.5}y^2 + \frac{3\times 1\times(-1)}{2^3\times 3!}(z^*)^{-1.5}y^3 + \cdots + \frac{3(-1)^n(2n-5)!\,(z^*)^{1.5-n}}{2^{2n-3}n!\,(n-3)!}y^n = \sum_{n=0}^{\infty}\alpha_n y^n \quad (19)$$

In the above equation, it is assumed that y = z – z*.

Changing the variable from z to y, Eq. (18) becomes:

$$\ddot{y} + \lambda^2\sum_{n=0}^{\infty}\alpha_n y^n = \varepsilon^2 a^2 V^2 sinVt \quad (20)$$

$$\ddot{y} + \sum_{n=0}^{\infty}\overline{\alpha_n}y^n = \varepsilon^2 a^2 V^2 sinVt \quad , \quad \overline{\alpha_n} = \lambda^2\alpha_n \quad (21)$$

In which αn are the coefficients of yn in the Taylor series of z1.5 about z*:

$$\alpha_1 = \frac{3}{2}(z^*)^{0.5}, \alpha_2 = \frac{3}{8}(z^*)^{-0.5}, \alpha_3 = -\frac{1}{16}(z^*)^{-1.5}, ... \quad (22)$$

From the definition, z* and F are related by the following equation:

$$F = m\lambda^2(z^*)^{1.5} \quad (23)$$

3. 2.3. Nonlinear Vibration: the Method of Multiple Scales

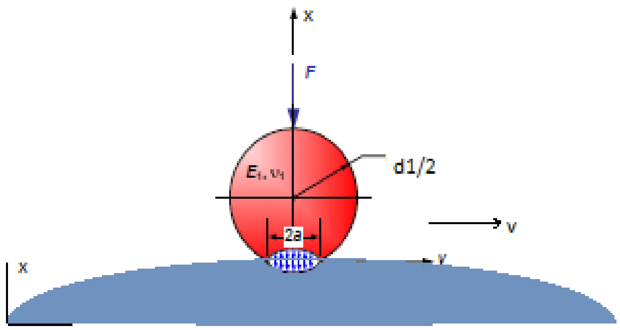

Fig2. wheel with velocity V on sinusoidal surface in contact by means of the compressive force

In order to implement the method of multiple scales to solve Eq. (21), new independent variables as follow, are introduced [11]:

$$T_n = \varepsilon^n t \quad for \quad n = 0,1,2,... \tag{24}$$

It follows that the derivatives with respect to t become expansions in terms of the partial derivatives with respect to the T_n according to

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \cdots = D_0 + \varepsilon D_1 + \cdots \tag{25}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_1) + \ldots$$

One assumes that the solution of Eq. (21) can be represented by an expansion having the form

$$y(t{:}\,\epsilon) = \epsilon y_1(T_0, T_1, T_2) + \epsilon^2 y_2(T_0, T_1, T_2) + \epsilon^3 y_3(T_0, T_1, T_2) + \cdots \tag{26}$$

Substituting (25) and (26) into (21) and equating the coefficients of ϵ, ϵ^2, and ϵ^3 to zero, we obtain:

$$\begin{aligned} &\epsilon^1; \quad D_0^2 y_1 + \omega_0^2 y_1 = 0 \\ &\epsilon^2; \quad D_0^2 y_2 + \omega_0^2 y_2 = -2D_0 D_1 y_1 - \bar{\alpha}_2 y_1^2 + a^2 V^2 sinVt \\ &\epsilon^3; \quad D_0^2 y_3 + \omega_0^2 y_3 = -2D_0 D_1 y_2 - {D_1}^2 y_1 - 2D_0 D_2 y_1 - 2\bar{\alpha}_2 y_1 y_2 - \bar{\alpha}_3 y_1^3 \end{aligned} \tag{27 (a-c)}$$

It is important to note that ω_0 is the natural frequency of the linearized system and is given by $\omega_0 = \sqrt{\alpha_1}$

After solving Eq. (27a) and substituting y_1 into Eq. (27b) we can find that:

$$y_1 = Ae^{i\omega_0 T_0} + \bar{A}e^{-i\omega_0 T_0} \tag{28}$$

$$y_2 = \frac{a^2 V^2}{{\omega_0}^2 - V^2} sinvt + \left[\frac{\bar{\alpha}_2 A^2}{3{\omega_0}^2} e^{2i\omega_0 T_0} - \frac{\bar{\alpha}_2}{{\omega_0}^2} A\bar{A} + CC\right] \tag{29}$$

where A is an unknown complex function and $\bar{A}$ is the complex conjugate of A and CC denotes the complex conjugate of the preceding terms.

Any particular solution of (27b) has a secular term containing the factor $-2D_1(Ae^{i\omega_0 T_0})$ unless $D_1(A) = 0$ Therefore A must be independent of T_1.

For calculation of y_3 we must substituting y_1 and y_2 into Eq. (27c) and recalling that $D_1(A) = 0$ we obtain

$$D_0^2 y_3 + \omega_0^2 y_3 = -\left[2i\omega_0 D_2 A - \frac{10\bar{\alpha}_2^2 - 9\bar{\alpha}_3\omega_0^2}{3\omega_0^2} A^2\bar{A} + \frac{2\bar{\alpha}_2 A a^2 V^2}{\omega_0^2 - V^2} sinVt\right] e^{i\omega_0 T_0} + \left[\frac{-3\bar{\alpha}_3\omega_0^2 - 2\bar{\alpha}_2^2}{3\omega_0^2}\right] A^3 e^{3i\omega_0 T_0} + CC \tag{30}$$

Considering that $(sinVT_0)e^{i\omega_0 T_0}$ is not a secular term, to eliminate secular terms from y3, we must put

$$2i\omega_0 D_2 A - \frac{10\bar{\alpha}_2^2 - 9\bar{\alpha}_3\omega_0^2}{3\omega_0^2} A^2\bar{A} = 0 \tag{31}$$

In solving equations having the form of (31), we find it convenient to write A in the polar form $A = \frac{1}{2}\gamma e^{i\beta}$

where γ and β are real functions of T_2. Substituting polar form of A into (31) and separating the result into real and imaginary parts, we obtain

$$\gamma'\omega_0 = 0 \tag{32}$$

$$\beta' = \frac{9\bar{\alpha}_3\omega_0^2 - 10\bar{\alpha}_2^2}{24\omega_0^2}\gamma^2$$

$$\beta = \left[\frac{9\bar{\alpha}_3\omega_0^2 - 10\bar{\alpha}_2^2}{24\omega_0^2}\gamma^2\right]\epsilon^2 t + \beta_0 \tag{33}$$

$$A = \frac{1}{2}\gamma \exp[i\left(\gamma^2\epsilon^2\frac{9\bar{\alpha}_3\omega_0^2 - 10\bar{\alpha}_2^2}{24\omega_0^2}\right)t + i\beta_0] \tag{34}$$

Substituting y_1 and y_2 from (28) and (29) into (26) and using (34), we obtain

$$y = \epsilon\gamma\cos(\omega t + \beta_0) - \frac{\epsilon^2\gamma^2\bar{\alpha}_2}{2\bar{\alpha}_1}\left[1 - \frac{1}{3}\cos(2\omega t + 2\beta_0)\right] + \frac{\epsilon^2 a^2 V^2}{{\omega_0}^2 - V^2} sinVt + O(\epsilon^3) \quad (35)$$

Where

$$\omega = \sqrt{\bar{\alpha}_1}\left[1 - \epsilon^2\gamma^2\frac{9\bar{\alpha}_3\bar{\alpha}_1 - 10\bar{\alpha}_2^2}{24{\bar{\alpha}_1}^2}\right] + O(\epsilon^3) \quad (36)$$

The indeterminate parameters in the above equations are $\epsilon\gamma$ and β_0, which are evaluated after applying the initial conditions.

We know that y=z-z* therefore z=y+z* and we have $x = z + \varepsilon^2 a^2 sinVt$. Substituting these relations into Eq. (35), the wheel displacement can be calculated

$$x = \epsilon\gamma\cos(\omega t + \beta_0) - \frac{\epsilon^2\gamma^2\bar{\alpha}_2}{2\bar{\alpha}_1}\left(1 - \frac{1}{3}\cos(2\omega t + 2\beta_0)\right) + \frac{\epsilon^2 a^2 V^2}{(\omega_0^2 - V^2)} sinVt + z^* + \epsilon^2 a^2 sinVt \quad (37)$$

2.4. Rail profile

Rail irregularities generally have a random distribution, and are considered as one of the major source of wagon vibration. The major causes of these irregularities are incompatible substrate conditions, whether conditions, rail age and excessive train commutation on rails [12].

From the experimental track geometry data collection, it is appear that the random profile of the rail track have standard deviation between 0.8 to 2.5 mm [7].

A sample function of rail irregularities can be generated numerically using the following series:

$$x = \sum_{n=1}^{\infty} a_n \cos(b_n V t) \quad (38)$$

For simulate a real rail profile the coefficient a_n and b_n is selected randomly such that the profile has standard deviation between 0.8 to 2.5 mm. Samples of the railway track irregularities is shown in figure 3.

Results and Discussion

To validate the method presented in this paper, response of the model using MMS method is compared with the result obtained from the Runge–Kutta numerical solution. The simulations are performed for sinusoidal and random irregularities.

Substituting the contact force into Eq. (6), the value of a is obtained. So, the value of C appearing in Eq. (9) can be evaluated as follows:

$$C = \left|\int_0^{\frac{d_1}{2a}}\left(\frac{-1}{1+{\tau_a}^2}\right)d\tau_a + 2v\int_0^{d_1/2a}\left(\left[1 - |\tau_a| tan^{-1}\left(\frac{1}{|\tau_a|}\right)\right] \times (1+\nu) - \frac{1}{2(1+{\tau_a}^2)}\right)d\tau_a\right| \quad (39)$$

The initial deflection of the center of the wheel resulted by the compressible force can be calculated as:

$$x^* = (F)^{\frac{2}{3}}\left(\frac{3C}{2\pi E K_a}\right) \quad (40)$$

The main parameters of the system used in the simulations are listed in Table 1.

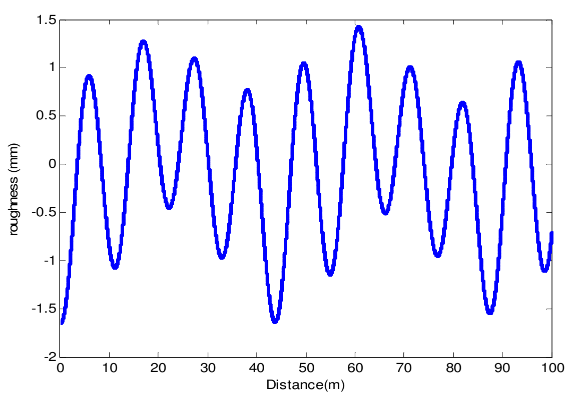

Fig3. Sample of the railway track irregularities

Table 1. Main parameters of rail and wheel

Parameters	Description	Value
E_r	Rail module of elasticity	$200\ Gpa$
E_w	Wheel module of elasticity	$200\ Gpa$
v_r	Rail poison's ratio	0.28
v_w	Wheel poison's ratio	0.28
m	Wheelset mass	$1200\ Kg$
x^*	Initial deflection of the wheel center	$3.23\ \mu m$

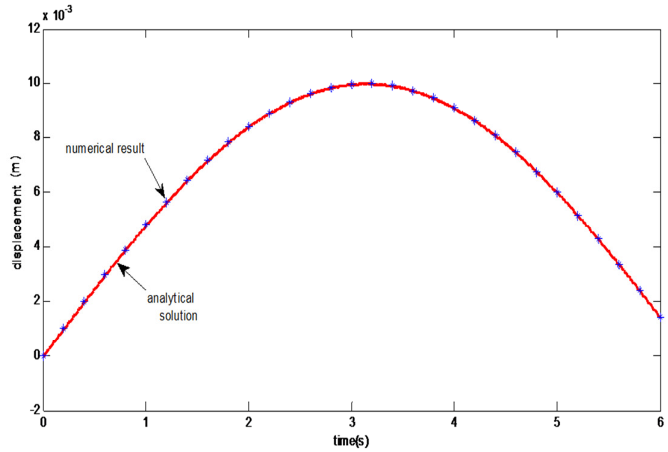

Fig4. Comparison between numerical and analytical time response of sphere's center

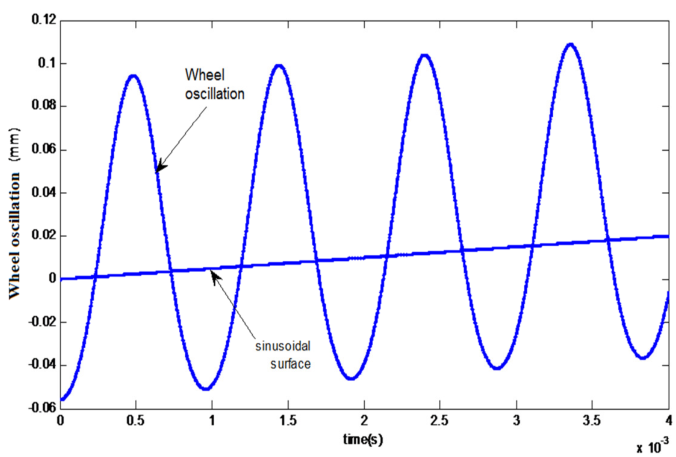

Fig5. oscillation of wheel with 0.5 m/s velocity on 1 cm sinusoidal irregularity amplitude

By determining mechanical and geometrical properties, K_a is evaluated from Eq. (7) as $K_a = 1.43 \times 10^{-4}$.

3.1. Sinusoidal rail irregularities

In this section, the simulation is carried out for a wheel moving on a straight track with sinusoidal irregularity. The dynamic response of the center of the wheel for v=0.5 m/s is presented in Figures 4 and 5. Result compared with the numerical solution obtained by the Runge–Kutta method. It is obvious that the analytical solution has a good agreement with the numerical solution.

Comparing the results presented in Figure 5 with sinusoidal surface shows that the wheel follows the sinusoidal irregularity according to Hertz elastic contact theory.

According to Hertz elastic contact theory the normal contact force between wheel and rail can be calculated. Using Eq. (11) the contact force has been calculated and illustrated in Figure 6.

Dependency of frequency on the preload applied to the wheel has been illustrated in figure 7. It can be seen that with increase in the preload, the natural frequency increases. This effect must be considered in the design, to put the working frequency away from the natural frequency of the system.

3.2. Random Rail Irregularities

The wheel set moving on a straight track with random irregularities has been simulated in this section. The displacement of the wheel set for V=100 Km/h is presented in Figures 8 and 9. Also, wheel/rail contact force on random profile is presented in Figure 10.

According to this figures the analytical solution presented in this paper for nonlinear wheel/rail contact problem can predict wheel displacement and wheel/rail contact force with high precision.

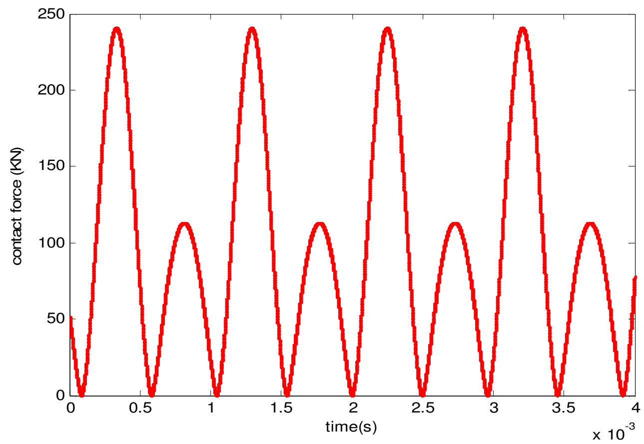

Fig6. contact force for wheel with 0.5 m/s velocity on 1 cm sinusoidal irregularity amplitude

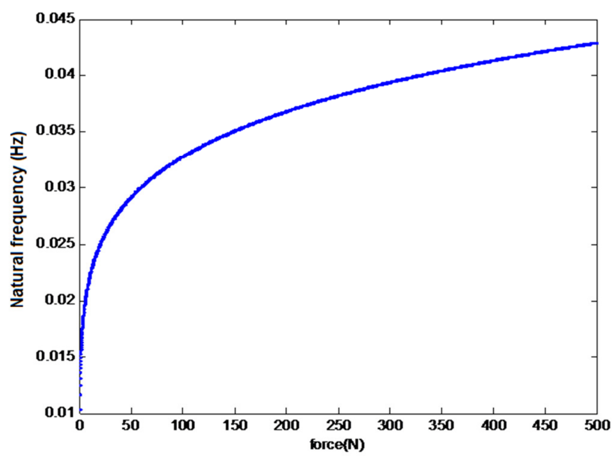

Fig7. Dependency of frequency on the preload

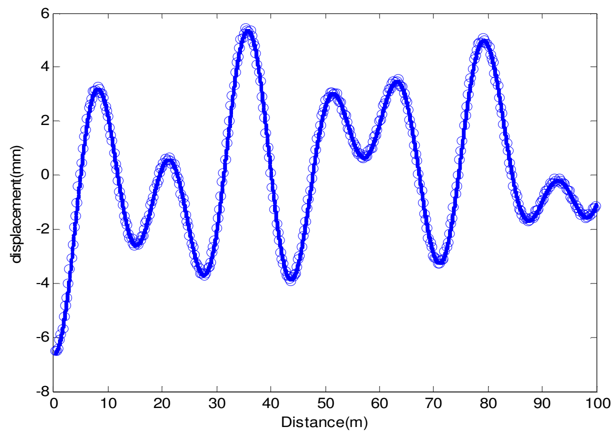

Fig8. displacement of wheel on random profile with velocity of 100 Km/h

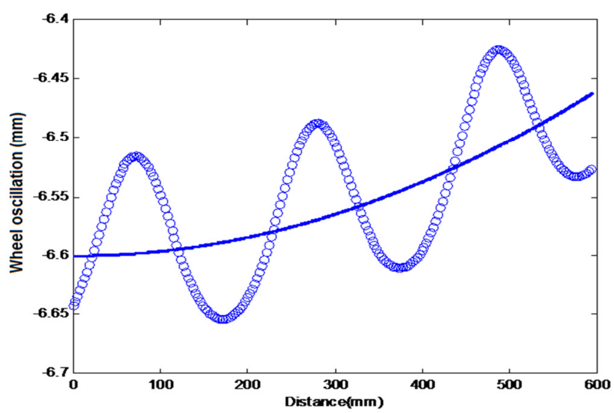

Fig9. oscillation of wheel on random profile (Zoom of figure 8)

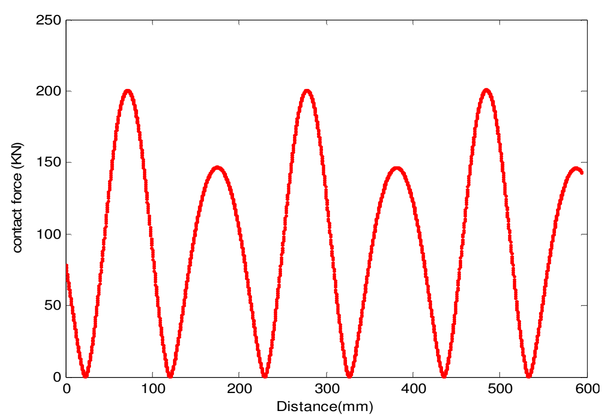

Fig10. contact force for wheel with velocity of 100 Km/h

Conclusion

In this paper, oscillation of a wheel on sinusoidal irregularities under a compressive force was studied. The Hertzian contact theory was used to obtain stress components in the wheel. By implementing the Method of Multiple Scale, frequency of vibration and the dynamic motion of the center of the wheel were obtained. Comparisons show that the results are very close to those extracted by the numerical methods.

Investigation on effects of preload on natural frequency has been shown that natural frequency increases by increasing the wheelset preload.

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Nomenclature

$\sigma_x, \sigma_y, \sigma_z$	Stress components in x, y and z directions
p	Contact pressure
F	Compressive force
x	Wheelset vertical displacement
$\epsilon_x, \epsilon_y, \epsilon_z$	Strain components in x, y and z directions
E	Wheel module of elasticity
ν	Wheel Poison ratio
δ	displacement of wheel's center
d_1	Wheel Diameter
R	Wheel radius
m	Wheelset Mass
V	Wheel velocity
t	time
ω_0	Natural frequency